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ON THE GEOMETRY OF INFINITE CYCLIC SUBGROUPS

BY

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ABSTRACT

In this paper we construct an example of a word metric on an infinite cyclic subgroup. This example shows that subexponential distortion does not obstruct non-trivial growth of connected radii. This answers a question of Gromov [6]. The constructed metric has other pathological properies. Specifically, its asymptotic cone depends on the choice of ultrafilter and scaling sequence.

1. Introduction

All groups under consideration are assumed to be finitely generated. If H is a group, l_H and d_H denote the word length and the word metric corresponding to some finite set of generators.

Consider spaces (X, d_X) and (Y, d_Y) such that $X \subset Y$. The notion of distortion describes the difference between intrinsic metric d_Y and $d_X|_Y$ on Y.

Definition 1: See [6]. Distortion function $\operatorname{dist}(r)$ is

$$\sup d_Y(y_1, y_2)/d_X(y_1, y_2)$$

where $d_X(y_1, y_2) \leq r$.

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We can use this definition for groups. If G is a subgroup of H we consider $X = (H, d_H)$ and $Y = (G, d_G)$, where d_H and d_G are the word metrics of H and G respectively. Distortion function depends on word metrics chosen. But if $\mathrm{dist}_1, \mathrm{dist}_2$ are distortion functions corresponding to different word metrics, then $\mathrm{dist}_1 / \mathrm{dist}_2$ is bounded for $r \geq 1$. In what follows we say that the distortion function dist is equivalent to f if dist/f and f/dist are bounded for $r \geq 1$.

Any reasonable function is equivalent in this sense to some distortion function ([8], [9]). It is known that if a subgroup of a hyperbolic group has subexponential distortion, then the distortion function is linear ([5]). For further examples of distortion see [1], [3], [6], [8], [9].

Suppose now that Y is k-connected. Fix in Y some point, say e. Consider the ball $B(e, R)$ in X. Its intersection with Y does not need to be k-connected.

Definition 2 (see [6], §4)*:* Connectedness radius $R_k(r)$ is $\inf L$, $L \geq r$ being such that any embedding of a l-sphere, $l \leq k$, into $B(e,r) \cap Y$ is contractible in $B(e, L) \cap Y$.

Now we want to apply this to the situation of $G \subset H$. A group itself is not k-connected, indeed it is not connected. So we have to take an appropriate thickening.

Definition 3: A metric space H_1 is a thickening of H if H is isometrically embedded into H_1 and $\sup d(h_1, H) < \infty$, $h_1 \in H_1$.

Definition 4 (see [6], §3)*:* H is large-scale k-connected, if for any thickening H_1 of H there exists a k-connected thickening H_2 of H_1.

For $k = 0$ as well as for geodesic H and $k = 1$ this definition becomes simpler, since in this case H is k-connected if and only if there exists a k-connected thickening of it. (For $k > 1$ the last condition is not sufficient.)

Note that any finitely generated group is large-scale 0-connected. The Cayley graph can be taken as a thickening to see this.

A group is 1-connected if and only if it is finitely presentable ([6], 1.C').

It is shown in [6], 4.A.1 that if the distortion function grows faster than any exponent then the growth of 0-connectedness radius is non-trivial. It was asked there (4.A.2) if the converse is true: that is, if at most exponential growth of distortion implies triviality of 0-connected radius.

We construct a counterexample to this. We show that an infinite cyclic subgroup of some finitely presented group can have at most quadratic distortion and non-trivial $R_0(R)$. Here non-trivial means that there is no constant C such that $R_0(R) \leq CR$.

2. Definition of the function l

In this section we construct (a length function of) some metric on $\mathbb{R}$. In the next section (Lemma 2) we check that this function is subadditive, and hence, in fact, defines a metric on a line. The main property of this metric is that it is "extremely unmonotone" (see the statement (4) of Lemma 1). This would imply the non-linearity of R_0. The statement (1) of Lemma 1 ensures that we can use a theorem of Ol'shanskii [8] and see that this metric is equivalent to some word metric on an infinite cyclic subgroup of some finitely generated group H. The same property implies that the distortion of this subgroup is at most exponential (moreover, it is at most quadratic.) The statement (3) of Lemma 1 allows us to use another theorem of Ol'shanskii [9] and to choose the group H to be finitely presented.

Define l: $\mathbb{R} \to \mathbb{R}_+$ as follows. Put $A^{(n)} = 10^{2*10^n}$. On the segment $[0, 10^{20}]$ put $l(x) = \sqrt{x}$. For $n \geq 1$ on the segment $[A^{(n)}, A^{(n+1)}]$ define l in the following way. Let

$$x_0^{(n)} = A^{(n)},$$
$$x_i^{(n)} = 2n^2 A^{(n)} i$$

for $1 \leq i \leq 2n$,

$$x_{2n+1}^{(n)} = 4n^4 A^{(n)}.$$

(In the following we drop the upper index n.) On $[x_0, x_{2n+1}]$ let l be piecewise linear function with l' having breaks at each x_i, such that

$$l(x_0) = \sqrt{A^{(n)}},$$

for $1 \leq i \leq n$

$$l(x_{2i}) = 2ni\sqrt{A^{(n)}},$$
$$l(x_{2i-1}) = 2n^2\sqrt{A^{(n)}},$$
$$l(x_{2n+1}) = 2n^2\sqrt{A^{(n)}}.$$

(Note that $l(x_{2n-1}) = l(x_{2n}) = l(x_{2n+1}) = \sqrt{x_{2n+1}}$.)

On the segment $[x_{2n+1}, A^{(n+1)}]$ let $l(x)$ be $\sqrt{x}$.

It remains to define $l(x)$ for $x < 0$, which we do by letting $l(x) = l(-x)$. See Figure 1 for the graph of l on the segment $[x_0, x_{2n}]$ in the case $n = 4$. The lower line is the graph of the square root. Note that the scale of the horizontal axis is much larger than that of the vertical one.

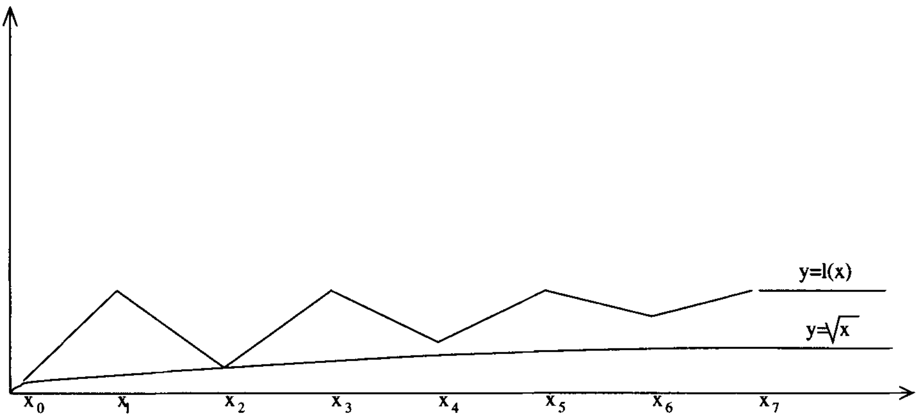

Figure 1.

3. Properties of l

LEMMA 1: *The constructed function l has the following properties.*

(1) $l(x) \geq \sqrt{\|x\|}$.
(2) $\max_{x\in[A^{(n)},A^{(n+1)}]} l(x) = l(A^{(n+1)})$ *and* $\min_{x\in[A^{(n)},A^{(n+1)}]} l(x) = l(A^{(n)})$.
(3) *The function $[l]|_{\mathbb{Z}}$ is computable.*
(4) $l(x_1^{(n)})/l(x_2^{(n)}) = n$.

Proof: (1) Since on the segment $[x_{2n-1}, x_{2n+1}]$ the function is constant and on $[x_{2n+1}, A^{(n+1)}]$ it is equal to $\sqrt{x}$, it is clear that $l(x) \geq \sqrt{x}$ on $[x_{2n-1}, A^{(n+1)}]$. Note that $l(x) \geq x/(n\sqrt{A^{(n)}}) \geq \sqrt{x}$ on $[x_2, x_{2n-1}]$. Note that $l(x) \geq \sqrt{x}$ on $[x_1, x_2]$, since l decreases on this segment and $l(x_2) = \sqrt{x_2}$. On $[x_0, x_1]$ we have $l(x) = x/\sqrt{x_0} \geq \sqrt{x}$.

(2) Note that l is non-decreasing on $[x_{2n}, A^{(n+1)}]$. Note also that $l(x_i) \geq l(x_0)$ for any i.

(3) This is clear from the definition of l.

(4) Note that $l(x_1^{(n)}) = 2n^2\sqrt{A^{(n)}}$ and $l(x_2^{(n)}) = 2n\sqrt{A^{(n)}}$.

LEMMA 2: *Function l is subadditive, that is, $l(a+b) \leq l(a) + l(b)$.*

Proof: Since for any x $l(x) = l(-x)$, it suffices to show that for $a, b \geq 0$

$$\|l(a+b) - l(b)\| \leq l(a).$$

In the following we will assume that $a, b \geq 0$. Let us prove the inequality by induction on n, assuming that $a + b \leq A^{(n+1)}$.

BASE: $n = 0$. Note that $\|\sqrt{a+b} - \sqrt{b}\| \leq \sqrt{a}$.

INDUCTION STEP: Assume that

$$\|l(a+b) - l(b)\| \leq l(a)$$

for $a + b \leq A^{(n)}$. Consider function f: $[0, A^{(n+1)}] \to \mathbb{R}$ such that $f(x) = l(x)$ for $x \in [A^{(n)}, A^{(n+1)}]$ and such that it is linear on $[0, A^{(n)}]$, $f(0) = 0, f(A^{(n)}) = \sqrt{A^{(n)}}$. Note that f has no break at $x_0^{(n)}$. Let us prove that

$$\|f(a+b) - f(b)\| \leq f(a).$$

(1) Let $a + b \leq x_{2n+1}$. Since f is piecewise linear on $[0, x_{2n+1}]$, it suffices to consider the cases in which at least two of the numbers $a, b, a+b$ are breaks of f'. As $x_1, \ldots, x_{2n}$ form an arithmetic progression, either all three numbers $a, b, a+b$ are breaks of f', or $a + b > x_{2n}$.

FIRST CASE: Suppose that $a + b \leq x_{2n}$. Then $a = x_k$, $b = x_m$, $a + b = x_{k+m}$. If both k and m are even, then $f(a+b) = f(a) + f(b)$. Otherwise, among $k, m, k+m$ there are two odd numbers. Hence two of $f(x_k), f(x_m), f(x_k + m)$ are equal to $2n^2\sqrt{A}$, and the third one is less than or equal to $2n^2\sqrt{A}$. So $f(x_k), f(x_m), f(x_{k+m})$ are sides of a triangle.

SECOND CASE: Suppose that $a + b > x_{2n}$. Then

$$f(a+b) = 2n^2\sqrt{A} \geq f(a), f(b).$$

If either $f(a)$ or $f(b)$ is equal to $2n^2\sqrt{A}$, then it is clear that $f(a), f(b), f(a+b)$ are sides of a triangle. If not, then $a = x_k, b = x_m$, k and m are even. Then $f(x) = x/(n\sqrt{A})$ for $x = a$, for $x = b$ and for $x = x_{2n}$. So $f(a+b) = f(x_{2n}) \leq f(a) + f(b)$.

(2) Now let us prove that

$$\|f(a+b) - f(b)\| \leq f(a)$$

for any positive a and b such that $a + b \leq A^{(n+1)}$. Now we can assume that $a + b > x_{2n+1}$. But then $f(a+b) \geq f(a), f(b)$ and

$$f(a+b) = \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \leq f(a) + f(b).$$

Now let us prove the induction step. Note that on $[0, A^{(n)}]$

$$f(x) = x/\sqrt{A^{(n)}} \leq \sqrt{x} \leq l(x).$$

Now consider a and b such that $a + b \leq A^{(n+1)}$. For $a, b \leq A^{(n)}$ we know the inequality by the induction hypothesis. Hence we assume that $a, b \geq A^{(n)}$. We know that $f(a), f(b), f(a+b)$ satisfy the triangle inequality. Note that $f(a+b) = l(a+b)$, $f(a) \leq l(a)$, $f(b) \leq l(b)$. If $f(a) < l(a)$, then $l(a) < l(A^{(n)})l(a+b)$, since $a < A^{(n)}$. Similarly, if $f(b) < l(b)$ then $l(b) < l(a+b)$. So the triangle inequality can only become stronger with passing from f to l.

Remark 1: Let $d(x, y) = l(y - x)$. Lemma 2 implies that l is a metric on $\mathbb{R}$.

Remark 2: For any $\alpha < 1$ one can similary construct l satisfying the conditions of Lemma 1 and Lemma 2, such that $l > x^{\alpha}$.

4. Non-trivial growth of connected radii

THEOREM 1: *There exists a finitely presented group H and an infinite cyclic subgroup $\mathbb{Z} \subset H$ such that its distortion in H satisfies $\operatorname{dist}(R) \leq AR^2$ for some $A > 0$, and for $l_H|_{\mathbb{Z}}$ the connected radius $\mathrm{R}_0(R)$ grows faster than linearly.*

Proof: First note that since l is subadditive and even, the function $\tilde{l}\colon \mathbb{Z} \to \mathbb{Z}_+$ defined by $\tilde{l}(z) = [l(z)] + 2$ for $z \neq 0$ and $\tilde{l}(0) = 0$ is also subadditive and even. Also note that $\tilde{l}$ is equivalent to l. Since $\tilde{l}(x) \geq l(x) \geq \sqrt{x}$, the number of integers n such that $\tilde{l}(n) < m$ is not greater than m^2. In particular, this number is subexponential and hence we can use a theorem of Ol'shanskii [9], [8] to see that there exists a finitely presented group H and a cyclic subgroup in H such that $l_H|_{\mathbb{Z}}$ is equivalent to $\tilde{l}$ (and hence to l). Obviously, the distortion function of the cyclic subgroup satisfies $\operatorname{dist}(r) \leq Ar^2$.

Note that for any $K > 0$ there exist $g, h \in \mathbb{N}$ such that $g < h$ and $l_H(g) > Kl_H(h)$. Suppose that $\mathrm{R}_0(R) \leq CR$. Let m be the maximal number such that $l_H(m) = 1$. There exist $0 < g < h$ such that

$$\frac{l_H(g)}{l_H(h)} > 2(C + m).$$

Consider $R = l(h)$. Note that e and h lie in different connected components of the intersection of $B(CR)$ with the Cayley graph of the cyclic subgroup. This contradiction proves the theorem.

Remark 3: Similarly, using Remark 2 one can see that for any $\alpha > 1$ there exists $\mathbb{Z} \subset H$ as in Theorem 1 such that $\mathrm{dist}(R) \le R^\alpha$.

Now we mention one more pathological property of the constructed word metric.

We recall the following definition. A **non-principal ultrafilter** Ω is a finitely additive measure on subsets of $\mathbb{N}$ such that each subset of $\mathbb{N}$ has measure either 0 or 1 and all finite subsets have measure 0. For any bounded function $h\colon \mathbb{N} \to \mathbb{R}$ its limit $h(\Omega)$ with respect to a non-principal ultrafilter Ω is uniquely defined by the following condition: for every $\epsilon > 0$

$$\Omega(\{i \in \mathbb{N} | \, |h(i) - h(\Omega)|| < \epsilon\}) = 1.$$

Let X be a metric space. Fix a sequence of scaling factors $\lambda_i \to \infty$, $\lambda_i \in \mathbb{R}$.

For any non-principal ultrafilter one can define an asymptotic cone $T = \mathrm{Con}_\Omega(X, \lambda_i)$ of X (for the definition see [6], [7]).

For many reasonable spaces (e.g., hyperbolic groups and spaces, nilpotent groups) the asymptotic cone does not depend on the choice of ultrafilter and scaling sequence.

Remark 4: Let $\mathbb{Z} \subset H$ be as in Theorem 1 and $Y = (\mathbb{Z}, d_H)$. Put

$$w_n = [A^{(n)}/n] = [10^{2*10^n}/n]$$

and consider $W = \{w_n | n \in \mathbb{N}\}$. Consider any ultrafilter Ω such that $\Omega(W) = 1$. Let $\varepsilon_n = \sqrt{n}$.

Now put $w'_n = A^{(n)}$ and $W' = \{w'_n | n \in \mathbb{N}\}$. Consider any ultrafilter Ω' such that $\Omega'(W') = 1$. Consider ε'_n such that $\varepsilon'_{A^{(n)}} = 2n^2\sqrt{A^{(n)}} = 2n^2 10^{10^n}$.

Then $\mathrm{Con}_\Omega(Y, \varepsilon_i)$ is homeomorphic to $\mathbb{R}$. But $\mathrm{Con}_{\Omega'}(Y, \varepsilon'_i)$ is not homeomorphic to $\mathbb{R}$, since it is easy to see that $([0, x_2^{(n)}], d_H/n)$ converges to an embedded loop in this space.

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